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A Perturbation Method for the Analysis of Wave Propagation in Inhomogeneous Dielectric Waveguides with Perturbed Media

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Abstract—**This paper presents a perturbation method for determining the modes and the propagation constants of TE and TM waves in inhomogeneous dielectric waveguides whose index distributions depart from well-known profiles; e.g., a parabolic profile for which exact solutions can be obtained. Applying the variable-transformation technique to the wave equations, the wave-equation problem is transformed into the related-equation problem. The approximate solutions of the wave equations are obtained solving the related equation. The method is applied to the analysis of lower order mode propagation in a near-parabolic-index medium. The first-order field functions and the second-order propagation constants are given.**

I. INTRODUCTION

THE PROBLEM of studying the behavior of electromagnetic waves in inhomogeneous media has been of great interest chiefly from mathematical and physical standpoints [1]–[6]. Later a number of methods [7]–[10] were developed to analyze this problem or the equivalent quantum-mechanics problem, most of which are based on the asymptotic expansion method [11] analogous to the Wentzel [3]–Kramers [6]–Brillouin [4], [5] (WKB) method, and these methods have been found to be very useful for weak inhomogeneities.

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Recently, a great variety of refractive-index distributions were used to realize self-focusing optical waveguides. Some of these distributions are not weakly inhomogeneous; the index variations within the distance of a wavelength are relatively rapid. In applying such media to single or quasi-single mode waveguides, it is necessary to analyze the propagation characteristics of lower order modes by suitable methods.

Kurtz and Streifer [8] have applied McKelvey's asymptotic method [7] to the problem of lower order mode propagation, and have found the solutions inaccurate near the center axis of the waveguide. Even if higher order asymptotic approaches are taken into account, it is impossible to improve the accuracy of the solutions near the center axis [11]. To avoid this defect, many authors [12] have used the variational method with the aid of a computer. However, computational labor will be required for the straightforward calculations [13], [14].

In this paper, an analytic method is presented to determine the transverse field functions and the propagation constants of TE and TM waves subjected to lower order mode propagation in inhomogeneous media. The method is based on two techniques. The one is the variable-transformation technique initially presented in nonuniform transmission-line problems by Berger [15] and later transferred to the equivalence problem of lenslike media by Yamamoto and Makimoto [16]. The other is the related equation technique developed in physics and mathematics by Wentzel [3], Gardner *et al.* [17], and McKelvey [7].

The wave equations for TE and TM waves are transformed into a related equation by introducing a new variable. The solutions of the wave equations are obtained solving the related equation with suitable approximations. The method is applied to the analysis of wave propagation in a near parabolic-index medium. The transverse field distributions and the propagation constants are calculated by first-order and second-order approximations, respectively. The results are discussed in comparison with those obtained by the WKB method and the variational method.

II. Wave Equations for TE and TM Waves

The inhomogeneous dielectric waveguide to be analyzed is two dimensional. The refractive index of the medium varies only in the transverse direction x, as $n(x)$; $n(x)$ is an even function of x. The waves propagate along the z direction and the field variation in the ζ direction is assumed to be uniform where (x,ζ,z) is a Cartesian coordinate system. The field components of TE and TM waves can be expressed in terms of the electric field E_ζ and the magnetic field H_ζ, respectively. The wave equations that E_ζ and H_ζ obey are [18]

$$\{\partial^2/\partial x^2 + \partial^2/\partial z^2 + k_0{}^2 n^2(x)\}E_\zeta = 0, \quad \text{for TE waves} \tag{1}$$

$$\{\partial^2/\partial x^2 + \partial^2/\partial z^2 + k_0{}^2 n^2(x) - n(x)[1/n(x)]''\} \cdot \{H_\zeta/n(x)\} = 0, \quad \text{for TM waves} \tag{2}$$

where k_0 is the wavenumber in free space and primes denote differentiation with respect to x.

We assume the phase variation $e^{-j\beta_{n1}z}$ for nth mode $(n = 0,1,2,\cdots)$ where β_{n1} is the propagation constant. Furthermore, we define

$$b_{n1} \equiv \begin{cases} 1 - (\beta_{n1}/k)^2, & \text{for TE waves} \\ 1 - (\beta_{n1}/k)^2 - (n_0/k^2)[1/n(x)]''_{x=0}, & \text{for TM waves} \end{cases} \tag{3}$$

$$\chi_1(x) \equiv \begin{cases} 1 - [n(x)/n_0]^2, & \text{for TE waves} \\ 1 - [n(x)/n_0]^2 + [n(x)/k^2][1/n(x)]'' \\ \quad - (n_0/k^2)[1/n(x)]''_{x=0}, & \text{for TM waves} \end{cases} \tag{4}$$

$$\Phi_{n1}(x)e^{-j\beta_{n1}z} \equiv \begin{cases} E_\zeta, & \text{for TE waves} \\ H_\zeta/n(x), & \text{for TM waves.} \end{cases} \tag{5}$$

Here, n_0 is the refractive index at the z axis $(= n(0))$, k is the wavenumber at the z axis $(= k_0 n_0)$, and $\chi_1(x)$ is a function of x satisfying $\chi_1(0) = 0$.

Using these quantities, both of the wave equations are written in the same form

$$\Phi_{n1}''(x) + k^2[b_{n1} - \chi_1(x)]\Phi_{n1}(x) = 0. \tag{6}$$

Hence we need only solve this simplified equation subject to the boundary condition at infinity; i.e., $\Phi_{n1}(\pm\infty) = 0$.

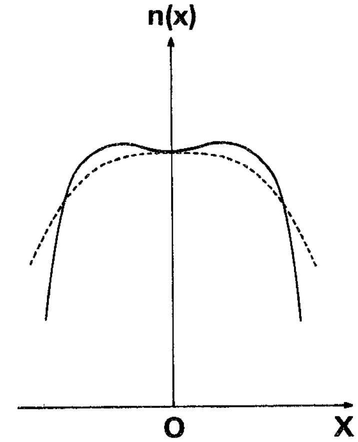

Fig. 1. Index profile. ——— Perturbed. - - - - - - - - Unperturbed.

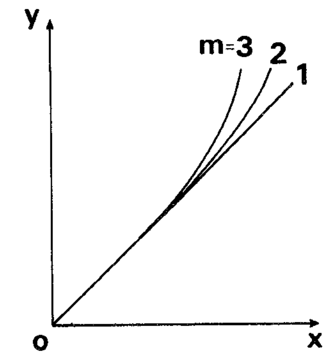

Fig. 2. A sketch of variable perturbation for $\chi_0(x) = (gx)^2$ and $\Delta\chi(x) = \alpha(gx)^{2m}$; α and g constants, $\Delta\chi(x) \equiv \chi_1(x) - \chi_0(x)$.

III. Variable Perturbation

For special cases when the refractive-index profile has, for instance, a parabolic shape or a step shape, the rigorous solutions of (6) can be obtained analytically. We now write such solutions as $\Phi_{n0}(x)$. The subscript 0 indicates that the index distribution is in the particular state where the propagation constants and the transverse field functions are known. This particular state is chosen to be an unperturbed state in the present perturbation problem (see Fig. 1). The unperturbed equation corresponding to (6) is

$$\Phi_{n0}''(x) + k^2[b_{n0} - \chi_0(x)]\Phi_{n0}(x) = 0. \tag{7}$$

For practical cases of interest, the oscillatory behaviors of the perturbed fields are probably similar to those of the unperturbed fields. We thus assume the perturbed field function $\Phi_{n1}(x)$ in the form[1]

$$\Phi_{n1}(x) \equiv [dy(x)/dx]^{-1/2}\Phi_{n0}[y(x)] \tag{8}$$

where the new variable y introduced is a function of x $(= y(x))$.

The aspect of $y(x)$ is sketched in Fig. 2; $y(x)$ is, in general, an odd function of x approximately proportional to x.

[1] The factor $(dy/dx)^{-1/2}$ was taken so that the first derivative of Φ_{n0} might be removed in (9).

The deviation of $y(x)$ from x is caused by the perturbation. To inspect such a deviation, we first substitute (8) into (6) using the differential formula

$$d^2/dx^2 + k^2[b_{n1} - \chi_1(x)] = (dx/dy)^{-2}\{\partial^2/\partial y^2 - [(d^2x/dy^2)/(dx/dy)]\,\partial/\partial y + k^2(dx/dy)^2[b_{n1} - \chi_1(x)]\}.$$

Then

$$\Phi_{n0}''(y) + \{x'''/2x' - 3(x''/x')^2/4 + k^2x'^2 \cdot [b_{n1} - \chi_1(x)]\}\Phi_{n0}(y) = 0 \quad (9)$$

where the primes denote differentiation with respect to y ($x = x(y)$). The first derivative of $\Phi_{n0}(y)$ does not appear in (9). This is the reason why we must place, in (8), the factor

$$(dy/dx)^{-1/2} \text{ in front of } \Phi_{n0}[y(x)].$$

In the second step, the variable x of (7) is merely replaced by y, and the resulting equation is compared with (9). Then we obtain the nonlinear differential equation

$$k^2\{b_{n0} - \chi_0[y(x)]\} = k^2x'^2\{b_{n1} - \chi_1(x)\} - \sqrt{x'}\,(1/\sqrt{x'})'' \quad (10)$$

where the relation $x'''/2x' - 3(x''/x')^2/4 = -\sqrt{x'}\,(1/\sqrt{x'})''$ is used. Alternatively, one may write (10) as

$$k^2\{b_{n0} - \chi_0[y(x)]\}y'^2 = k^2\{b_{n1} - \chi_1(x)\} + \sqrt{y'}\,(1/\sqrt{y'})''. \quad (11)$$

This is the related equation to be solved instead of the wave equations.

In the next section, we present several approaches for (11). For the sake of clarity, the application is restricted to a near parabolic type of profile. The formulation presented there is, however, applicable to the general classes of index distribution.

IV. Application

We apply the method to the near parabolic-index medium

$$n(x) = n_0\sqrt{1 - \delta\left(\frac{x}{a}\right)^2 - \alpha\delta^2\left(\frac{x}{a}\right)^4} \quad (12)$$

where δ, a, and α are constants.

From (3), the propagation constants are defined as

$$\beta_{n1} \equiv \begin{cases} k\sqrt{1 - b_{n1}}, & \text{for TE waves} \\ k\sqrt{1 - b_{n1} - \delta/(ka)^2}, & \text{for TM waves.} \end{cases} \quad (13)$$

Also from (4)

$$\chi_1(x) \equiv (gx)^2 + \alpha(gx)^4 \quad (14)$$

where g is defined as

$$g \equiv \begin{cases} \dfrac{\sqrt{\delta}}{a}, & \text{for TE waves} \\ \dfrac{\sqrt{\delta}}{a}\cdot\sqrt{1 + \dfrac{6\alpha + 4}{(ka)^2}\delta}, & \text{for TM waves.} \end{cases} \quad (15)$$

Note that (14) holds exactly for TE waves and approximately for TM waves.

The unperturbed state in this case is the parabolic-index distribution

$$\chi_0(x) = (gx)^2. \quad (16)$$

The unperturbed solutions are [19]

$$b_{n0} = (g/k)(2n + 1)$$

$$\Phi_{n0}(x) = \exp\left(-\frac{1}{2}\frac{x^2}{S_0{}^2}\right) H_n\left(\frac{x}{S_0}\right), \qquad S_0 \equiv 1/\sqrt{kg} \quad (17)$$

where H_n is a Hermite polynomial of nth degree. The perturbed solutions are determined only by inserting the solution of the related equation into (8) and using (17).

A. First-Order Solutions of (11)

To solve the nonlinear differential equation (11) under the first-order approximation, we introduce the first-order quantities

$$\begin{aligned} \Delta b_n &\equiv b_{n1} - b_{n0} \\ \Delta y(x) &\equiv y(x) - x \\ \Delta\chi(x) &\equiv \chi_1(x) - \chi_0(x) \\ &= \alpha(gx)^4. \end{aligned} \quad (18)$$

Equation (11) is expressed in terms of (18) and is approximated by the first-order equation

$$\frac{1}{2k^2}\Delta y'''(x) + 2[b_{n0} - \chi_0(x)]\,\Delta y'(x) - \chi_0'(x)\,\Delta y(x) = \Delta b_n - \Delta\chi(x). \quad (19)$$

It is possible mathematically to obtain the rigorous solutions of (19) even if the unperturbed state is chosen arbitrary. The detailed derivation is given in Appendix I. The results are

$$\Delta b_n = \int_{-\infty}^{\infty} \Delta\chi(x)f_n(x)\,dx \Big/ \int_{-\infty}^{\infty} f_n(x)\,dx = \left(\frac{g}{k}\right)^2 \alpha\,\frac{3}{4}(2n^2 + 2n + 1) \quad (20)$$

$$\Delta y(x) = 2k^2\int_0^x dt\,\frac{f_n(x)}{f_n(t)}\left\{\int_0^t [\Delta b_n - \Delta\chi(s)]f_n(s)\,ds\right\} \cdot \left\{\int_t^x \frac{ds}{f_n(s)}\right\}, \qquad (f_n(x) \equiv \Phi_{n0}{}^2(x)). \quad (21)$$

It must be pointed out that, although the present method differs from the variational method, the first-order approach in this section yields the same result for Δb_n [see (20)].

Calculation of (21) is performed integrating by parts. This is tedious, but the results are simple as listed in Table I. A plot of the normalized values ($h_n \equiv k\,\Delta y(x)/S_0 g\alpha$) versus x/S_0 ($\equiv \xi$) is in Fig. 3. The transverse field distributions computed for $\alpha = 500$, $g/k = 2 \times 10^{-4}$ (see Fig. 4) are plotted in Fig. 5 (a)–(d) (solid curves). The short-dash curves are unperturbed field distributions ($\alpha = 0$). For

TABLE I
FUNCTIONS $h_n(\xi) \equiv k\,\Delta y(x)/S_0 g\alpha$; $\xi \equiv x/S_0$

$h_0(\xi)$	$\frac{1}{8}\xi^3 + \frac{3}{16}\xi - \frac{3}{16}F(\xi)$
$h_1(\xi)$	$\frac{1}{8}\xi^3 + \frac{9}{8}\xi^2 F(\xi)$
$h_2(\xi)$	$\frac{17}{16}\xi^3 + \frac{15}{32}\xi - \frac{15}{32}(2\xi^2-1)^2 F(\xi)$
$h_3(\xi)$	$-\frac{7}{8}\xi^5 + \frac{37}{16}\xi^3 + \frac{7}{16}\xi^2(2\xi^2-3)^2 F(\xi)$
$h_4(\xi)$	$\frac{9}{16}\xi^7 - \frac{99}{32}\xi^5 + \frac{305}{64}\xi^3 + \frac{81}{128}\xi - \frac{9}{128}(4\xi^4-12\xi^2+3)^2 F(\xi)$
$h_5(\xi)$	$-\frac{11}{40}\xi^9 + \frac{209}{80}\xi^7 - \frac{1243}{160}\xi^5 + \frac{481}{64}\xi^3 + \frac{11}{320}\xi^2(4\xi^4-20\xi^2+15)^2 F(\xi)$

$$F(\xi) = e^{-\xi^2}\int_0^{\xi} e^{t^2}\,dt = \xi - \frac{2}{3}\xi^3 + \frac{2^2}{5\cdot 3}\xi^5 - \frac{2^3}{7\cdot 5\cdot 3}\xi^7 \cdots\cdots$$
$$= \frac{1}{2\xi} + \frac{1}{4\xi^3} + \frac{3}{8\xi^5} + \frac{15}{16\xi^7} \cdots\cdots$$

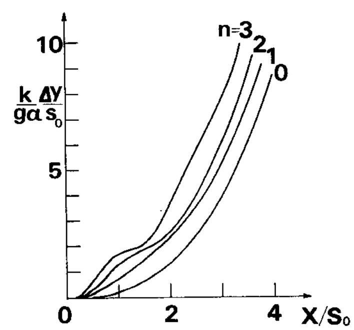

Fig. 3. A plot of $h_n(\xi) \equiv (k/g\alpha)(\Delta y/S_0)$; $\xi \equiv x/S_0$.

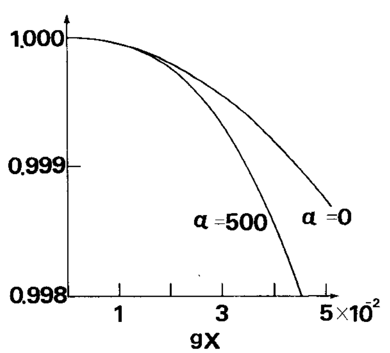

Fig. 4. Normalized parabolic-index distribution with fourth aberration. $\alpha = 0$ unperturbed.

comparison, the power-series solutions[2] are given in the figure (dot–long-dash curves).

B. Second-Order Formula for Δb_n

The first-order equation (19) can be extended to the second-order expression in a similar manner. Such an expression (Appendix II) is available for calculation of the second-order values of Δb_n. To explain the procedure of calculation, we use the notation $\Delta\chi_{\text{correct}}(x)$ defined by (A4).

As is seen from the definition, $\Delta\chi_{\text{correct}}(x)$ is the corrective term that involves the first-order solution of $\Delta y(x)$ and the first-order value of Δb_n. Thus $\Delta\chi_{\text{correct}}(x)$ is a known quantity.

The second-order equation for $\Delta y(x)$ [see (A3)] has the same form as (19) if $\Delta\chi(x)$ in (19) is replaced by $\Delta\chi(x) - \Delta\chi_{\text{correct}}(x)$. This means that the previous procedure developed in the first-order analysis can be applied to the present case by this replacement. When this is done in (20), we obtain the second-order formula for Δb_n

$$\Delta b_n = \Bigg\{ \int_{-\infty}^{\infty} \Delta\chi(x) f_n(x)\,dx - \int_{-\infty}^{\infty} \bigg[\frac{3}{4k^2}(\Delta y'')^2 + \tfrac{1}{2}\chi_0''(x)(\Delta y)^2 - 3[b_{n0} - \chi_0(x)](\Delta y')^2 + 3\chi_0'(x)\,\Delta y\,\Delta y' + [\Delta b_n - \Delta\chi(x)]\,\Delta y' \bigg] \cdot f_n(x)\,dx \Bigg\} \Bigg/ \int_{-\infty}^{\infty} f_n(x)\,dx \tag{22}$$

where $\Delta\chi_{\text{correct}}(x)$ is rewritten in terms of the first-order solution of $\Delta y(x)$.

A manipulation is made to simplify the calculation of (22) (Appendix III). As a result, we obtain

$$\Delta b_n = \left(\frac{g}{k}\right)^2 \alpha \frac{3}{4}(2n^2 + 2n + 1) - \left(\frac{g}{k}\right)^3 \alpha^2 \cdot \left[\frac{17}{64}(2n+1)^3 + \frac{67}{64}(2n+1)\right]. \tag{23}$$

The first part of (23) is the previous result obtained by the first-order approximation and the second part the corrective

[2] The power-series solutions were obtained from the wave equation by means of expansion, initially giving the first-order approximate value of Δb_0 for a fundamental mode, and the WKB values of Δb_n for higher order modes. For more precise treatment, the reader is referred to Dil and Blok [20].

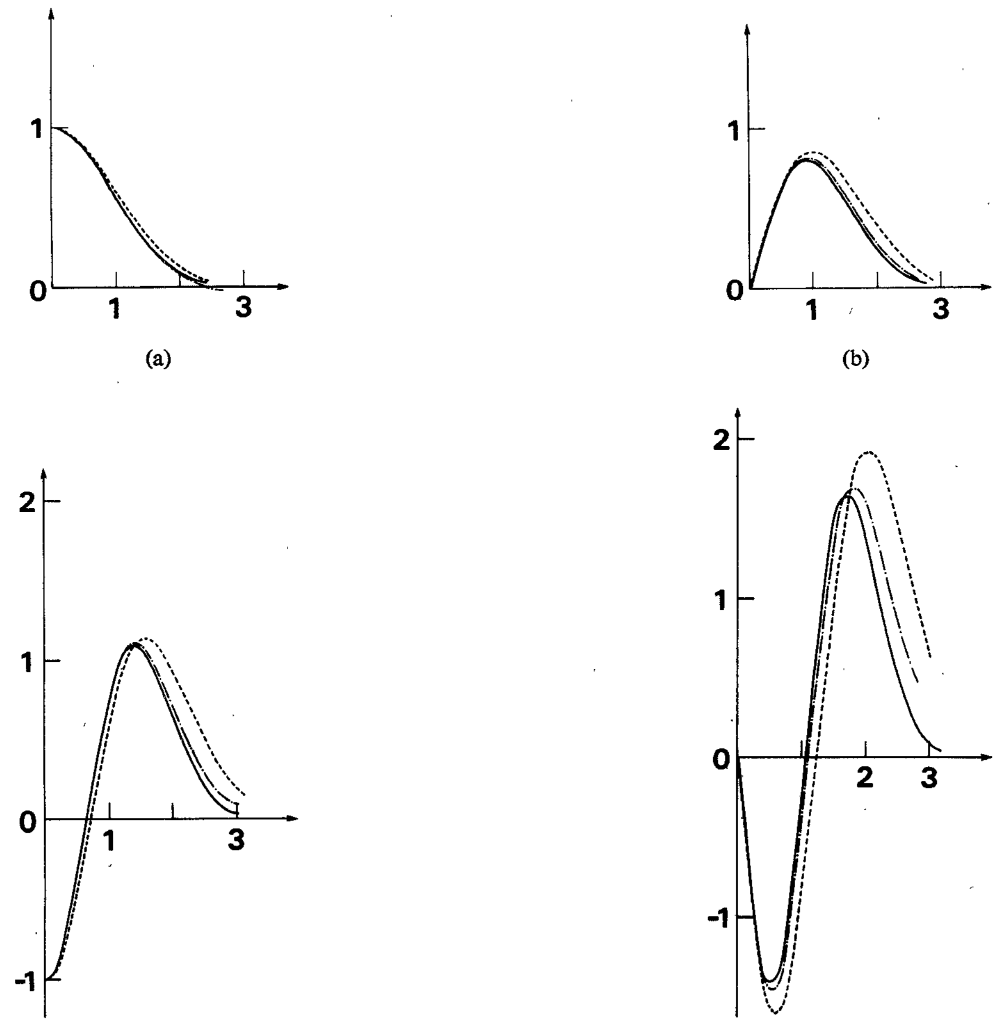

Fig. 5. Field distributions versus x/S_0. -------- Unperturbed. ——— Perturbed. — · — · — Power-series solutions. (a) Fundamental mode ($n = 0$). (b) Second mode ($n = 1$). (c) Third mode ($n = 2$). (d) Fourth mode ($n = 3$).

TABLE II
VALUES OF $\kappa_n \equiv (k/g)^2 \Delta b_n/\alpha$ FOR $g\alpha/k = 0.1$

n	0	1	2	3
2nd order (WKB)	0.3484	2.658	6.055	9.264
Exact (WKB)	0.3517	2.854	7.306	13.372
1st order (V.M)	0.7500	3.750	9.750	18.750
2nd order (Eq. (23))	0.6188	2.719	5.906	8.906

term. To check the validity of (23) for large n, Δb_n is calculated using the WKB method [refer to (29)] by the second-order approximation

$$\Delta b_n(\text{WKB}) = \left(\frac{g}{k}\right)^2 \alpha \frac{3}{4}(2n^2 + 2n + \tfrac{1}{2}) - \left(\frac{g}{k}\right)^3 \alpha^2 \frac{17}{64}(2n + 1)^3. \quad (24)$$

Obviously, both agree for higher order modes, but not for lower order modes. The normalized values $\kappa_n \equiv (k/g)^2 \Delta b_n/\alpha$ derived from various methods are compared in Table II.

C. Other Approximations

Before ending this section, we present the other approaches for (11) to clarify the features of this method.

1) $\Delta y'''(x)/2k^2 \simeq 0$ *in (19):* This assumption is reasonable when the index variation is sufficiently smooth. Then the solution of (19) is

$$\Delta y(x) = \frac{1}{2\sqrt{b_{n0} - \chi_0(x)}} \int_0^x \frac{\Delta b_n - \Delta\chi(t)}{\sqrt{b_{n0} - \chi_0(t)}}\, dt$$
$$= \left(\frac{g}{k}\right) \alpha S_0 \left[\frac{1}{8}\left(\frac{x}{S_0}\right)^3 + \frac{3}{16}(2n+1)\left(\frac{x}{S_0}\right)\right]. \quad (25)$$

The value of $\Delta y(x)$ must be finite at the turning point x_T (the point such that $b_{n0} - \chi_0(x_T) = 0$), whereas the factor $(b_{n0} - \chi_0(x))^{-1/2}/2$ on the right-hand side of (25) becomes infinite at x_T. This means that the value of the integral in (25) must vanish at x_T. Namely,

$$\Delta b_n = \int_0^{x_T} \frac{\Delta\chi(t)}{\sqrt{b_{n0} - \chi_0(t)}}\, dt \bigg/ \int_0^{x_T} \frac{dt}{\sqrt{b_{n0} - \chi_0(t)}}$$
$$= \frac{3}{4} \alpha \left(\frac{g}{k}\right)^2 (2n^2 + 2n + \tfrac{1}{2}). \quad (26)$$

It is remarkable that (26) agrees with the first-order result derived by the WKB method [refer to (24)]. It can also be proved that (21) tends to (25) as x increases. Therefore, for higher order modes, the first-order results (20) and (21) agree well with (26) and (25), respectively.

2) $\sqrt{y'}\,(1/\sqrt{y'})'' \simeq 0$ *in (11):* This assumption is also reasonable for weak inhomogeneities. Then (11) can be written as

$$\int_0^x \sqrt{b_{n1} - \chi_1(x)}\, dx$$
$$= \int_0^{y(x)} \sqrt{b_{n0} - \chi_0(y)}\, dy$$
$$= \sqrt{\frac{g}{k}(2n+1)} \int_0^{y(x)} \sqrt{1 - y^2/y_T^2}\, dy \quad (27)$$

where y_T denotes the turning point $(= S_0\sqrt{2n+1})$ [10], [19] of the parabolic-index medium. Equation (27) must be satisfied over the entire region including the oscillatory region $|y| < y_T$ and the damped region $|y| > y_T$. Use of $dy/dx = \sqrt{b_{n1} - \chi_1(x)}/\sqrt{b_{n0} - \chi_0[y(x)]}$ leads (8) to

$$\Phi_{n1}(x) = \left[\frac{b_{n0} - \chi_0[y(x)]}{b_{n1} - \chi_1(x)}\right]^{1/4} \exp\left[-\frac{1}{2}\frac{y^2(x)}{S_0^2}\right] H_n\left[\frac{y(x)}{S_0}\right]. \quad (28)$$

The propagation constants are determined by setting $x = x_T$ and $y = y_T$ in (27) where x_T is the turning point of the perturbed medium $(b_{n1} - \chi_1(x_T) = 0)$

$$\int_0^{x_T} \sqrt{b_{n1} - \chi_1(x)}\, dx = \int_0^{y_T} \sqrt{b_{n0} - \chi_0(y)}\, dy$$
$$= \frac{\pi}{4k}(2n+1). \quad (29)$$

This is the Bohr–Sommerfeld condition [11] formulated by Kramers [6]. Since $y(x)$ and x are related under the condition (29) by (27), the term $[b_{n0} - \chi_0[y(x)]]/[b_{n1} - \chi_1(x)]$ does not diverge at the turning point; there is no singularity in (28). Hence (28) is applicable over the entire region. Note that the range of application for the WKB asymptotic solution is restricted to the oscillatory region [11].

V. Discussions

The problems of wave propagation for TE and TM waves have been dealt with simultaneously. Both solutions have been obtained within the same accuracy. The propagation constants of TM waves calculated from (23) include the first-order result obtained by Marcuse [19] and Ghatak and Kraus [18].

For general index distributions, it is possible to make use of the formulation presented in Section IV. The unperturbed state can also be chosen arbitrary under the assumption of even index distribution, as mentioned before.

Higher order approaches may be available for more accurate analyses in different ways as follows.

1) Retain the unperturbed state and solve the related equation iteratively as in the second-order derivation for Δb_n.

2) Insert the first-order result just obtained into the unperturbed state, calculate the second-order solutions, and repeat this iterative procedure.

It is worth mentioning that although this paper is not concerned with the wave propagation of higher order modes, the field expression (28) just derived gives the advantage of plotting the field distributions of such modes in comparison with the asymptotic expressions [3]–[8] obtained by the conventional asymptotic methods. The propagation constants, however, are similar to those derived by asymptotic expansion. Therefore, the present method does not have the advantage of obtaining more accurate results for propagation constants of higher order modes.

VI. Conclusion

A perturbation method has been proposed to determine the propagation constants and the field distributions of lower order modes in inhomogeneous media. Compared to current methods, the proposed method is useful for: 1) relatively strong perturbation; 2) obtaining analytic solutions; and 3) plotting field distributions over the entire region.

The problems of TE and TM wave propagations in a near parabolic-index medium have been solved. The first-order field functions and the second-order propagation constants have been obtained.

Appendix I
Derivation of (20) and (21)

Multiplying both sides of (19) by $f_n(x)$ and integrating by parts over the interval $[-\infty,\infty]$ results in

$$\int_{-\infty}^{\infty} [\Delta b_n - \Delta\chi(x)] f_n(x)\, dx$$
$$= \frac{1}{2k^2} [f_n\, \Delta y'' + f_n''\, \Delta y - f_n'\, \Delta y' + 4k^2(b_{n0} - \chi_0) f_n\, \Delta y]_{-\infty}^{\infty}, \quad (f_n(x) \equiv \Phi_{n0}^2(x)) \quad (A1)$$

where the relation $f_n'''/2k^2 = -2(n_{n0} - \chi_0)f_n' + \chi_0' f_n$ is used. The Δy will increase at most with algebraic growth, while f_n decreases exponentially. Hence, the right-hand side of (A1) vanishes. This equation is alternatively written as (20).

The (21) was obtained by trial; the validity was proved by straightforward substitution into (19). The (21) satisfies the conditions

1) An odd function of x

2) $\Delta y(x) \to 0(x^3)$ as $x \to 0$. (A2)

The first is required from a physical standpoint. The second is the normalization condition with which the magnitude of the perturbed field function is normalized at the origin ($x = 0$) to that of the unperturbed one.

Another possible derivation of (21) is to use the Wronskian technique [20] developed for ordinary differential equations. A detailed and instructive interpretation for this technique is given in [20]. According to the Wronskian analysis, the solution satisfying (A2) can be written in the form

$$\Delta y(x) = \int_0^x G_n(x,t)[\Delta b_n - \Delta\chi(t)]\, dt \qquad \text{(A2.1)}$$

where G_n is the one-sided Green function

$$G_n(x,t) = k^2 f_n(x) f_n(t) \left(\int_t^x \frac{ds}{f_n(s)} \right)^2.$$

It can easily be proved that (21) is equivalent to (A.2.1) when Δb_n is given by (20) (the proof is omitted here).

Appendix II

Second-Order Equation of (11)

Use $\sqrt{y'}\,(1/\sqrt{y'})'' \simeq -\Delta y'''/2 + \Delta y'\,\Delta y'''/2 + 3(\Delta y'')^2/4$ and $y'^2 = 1 + 2\,\Delta y' + (\Delta y')^2$. Then we obtain

$$\frac{1}{2k^2}\Delta y'''(x) + 2[b_{n0} - \chi_0(x)]\,\Delta y'(x) - \chi_0'(x)\,\Delta y(x) = \Delta b_n - \Delta\chi(x) + \Delta\chi_{\text{correct}}(x) \qquad \text{(A3)}$$

where $\Delta\chi_{\text{correct}}(x)$ is the second-order corrective term

$$\Delta\chi_{\text{correct}}(x) = \tfrac{1}{2}\chi_0''(x)(\Delta y)^2 + 2\chi_0'(x)\,\Delta y\,\Delta y' - [b_{n0} - \chi_0(x)](\Delta y')^2 + \frac{1}{2k^2}(\Delta y\,\Delta y''' + \tfrac{3}{2}\,\Delta y''^2).$$

The first-order solution of Δy obtained is substituted into $\Delta\chi_{\text{correct}}(x)$. Hence, using (19), we can rewrite

$$\Delta\chi_{\text{correct}}(x) = \frac{3}{4k^2}(\Delta y'')^2 + \tfrac{1}{2}\chi_0''(x)(\Delta y)^2 - 3[b_{n0} - \chi_0(x)](\Delta y')^2 + 3\chi_0'(x)\,\Delta y\,\Delta y' + [\Delta b_n - \Delta\chi(x)]\,\Delta y'. \qquad \text{(A4)}$$

Appendix III

Calculation of (22)

The first-order solutions (21) are too complicated to be inserted into (22). Instead, we used

$$\Delta y(x) = \left(\frac{g}{k}\right)\alpha S_0 \left[\frac{1}{8}\left(\frac{x}{S_0}\right)^3 + \frac{3}{16}(2n+1)\left(\frac{x}{S_0}\right)\right]. \qquad \text{(A5)}$$

Although (A5) does not satisfy the second condition of (A2), (A5) is also a solution of (19). As mentioned in Appendix I, the second condition is not essential for calculation of (22), because the propagation constants are independent of the normalization of field magnitude. Therefore, use of (A5) for (22) leads to a correct result.

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Slow-Wave Propagation Along Variable Schottky-Contact Microstrip Line

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Abstract—Schottky-contact microstrip lines (SCML) are a special type of transmission line on the semiconducting substrate: the metallic-strip conductor is specially selected to form a rectifying metal–semiconductor transition while the ground plane exhibits an ohmic metallization. Thus the cross section of SCML is similar to that of a Schottky-barrier diode. The resulting voltage-dependent capacitance per unit length causes the nonlinear behavior of such lines.

In this paper a detailed analysis of the slow-wave propagation on SCML is presented, including the effect of metallic losses. Formulas for the propagation constant and characteristic impedance are derived and an equivalent circuit is presented. Conditions for slow-mode behavior are given, particularly taking into account the influence of imperfect conductors and defining the range of many interesting applications. Experimental results performed on Si-SCML are compared with theory.

I. Introduction

DUE to particular applications in microwave integrated circuits, microstrip lines on semiconductor substrates have been thoroughly investigated both in theoretical and experimental works during the last few years. The Schottky-contact microstrip line (SCML) is a special form of microstrip line on a semiconducting substrate: the cross section [Fig. 1(a)] is similar to a Schottky-barrier diode; i.e., the stripline forms a rectifying metal–semiconductor transition to the chip with a large-area ohmic-contact back metallization. At the Schottky-barrier contact a depletion layer arises, the depth of which depends strongly on the applied voltage. Thus the most interesting features of SCML's are caused by this voltage-dependent depletion-layer capacitance per unit length. Two modes of operation may be distinguished: the large-signal behavior, which is characterized by nonlinear wave propagation, and the small-signal properties, which are determined by bias-dependent transmission-line parameters.

The wave propagation on SCML's has been investigated recently, leading to several fundamental results: large-signal operation leads to distributed harmonic frequency generation and the possibility of parametric amplification [1], [2]. Under small-signal conditions a slow-wave propagation occurs, and the propagation constant and characteristic impedance may be changed by an external dc bias [3], [4]. In particular, it has been verified that bias-dependent phase delay gives rise to possible applications of SCML in variable IC microwave components, such as resonators, delay lines, phase shifters, or tunable filters [5]–[7].

To a certain extent the SCML resembles the microstrip line, which serves as the electrical-interconnection pattern in IC technology on MOS or MIS systems where an oxide layer insulates the semiconductor wafer from the metallic conductors. The high-frequency behavior has been investigated by several workers, since the propagation delay imposes a limitation upon signal velocity [8]–[11]. Introducing the voltage-dependent capacitance of the MIS system, a variable (nonlinear) MIS microstrip line results [12], [13]. The fundamental theoretical work on wave propagation along such transmission lines has been done by Guckel *et al.* [8], assuming perfect conductors and a large ratio $r = w/l$ of strip width w to substrate thickness l.

Until now an accurate calculation of the influence of imperfect metallic conductors has been neglected in theoretical analysis. The experimental results, however, show large deviations from theory, especially in the lower slow-mode region [4], [11] which exhibits the most interesting features for possible applications. The efficiency of harmonic-frequency conversion and parametric amplification in nonlinear SCML depends strongly on the metallic losses [1], [2], and the phase delay of the variable SCML is influenced by the additional attenuation. In this way, the influence of the metallic losses has become a central problem in the discussion of possible practical applications.

In this paper, a more detailed analysis of small-signal slow-wave propagation along variable SCML is presented, including the effect of imperfect conductors. The treatment

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